Math 8 Homework 8

1. Suppose that $x_1, \ldots, x_n$ are real numbers such that
$$x_1 + 2x_2 + 3x_3 + \cdots + nx_n = 1.$$
Find the minimum possible value of $x_1^2 + x_2^2 + \cdots + x_n^2$ with proof.

2. Suppose that $x, y, z > 0$ and that $x + y + z = 1$. Show that
$$\frac{1}{x} + \frac{4}{y} + \frac{9}{z} \geq 36.$$

3. Suppose that $a, b > 0$ and $a + b = 1$. Show that
$$\left(a + \frac{1}{a}\right)^2 + \left(b + \frac{1}{b}\right)^2 \geq \frac{25}{2}.$$

4. Prove there does not exist a sequence $a_1, a_2, a_3, \ldots$ of reals so that for all $m \in \mathbb{N}$ we have
$$a_1^m + a_2^m + a_3^m + \cdots = m.$$

5. Consider n arbitrary real numbers $x_1, x_2, \ldots, x_n$.
   (a) Prove Carlson's inequality:
$$(x_1 + x_2 + \cdots + x_n)^2 < \frac{\pi^2}{6}(x_1^2 + 4x_2^2 + 9x_3^2 + \cdots + n^2 x_n^2).$$
   (b) Show that $\pi^2/6$ cannot be replaced by any smaller constant if we allow n to be arbitrarily large.

6. Show that for positive reals a, b, c with $a + b + c = 1$
$$\frac{a}{a^2+1} + \frac{b}{b^2+1} + \frac{c}{c^2+1} \leq \frac{9}{10}.$$

7. Given positive numbers a, b, c, prove that
$$a^a b^b c^c \geq (abc)^{(a+b+c)/3}.$$

8. Consider an acute triangle ABC with side lengths a, b, c and area K. Prove each of the following.
   (a) $\sec^2 A + \sec^2 B + \sec^2 C \geq 12$
   (b) $\sin A \sin B \sin C \leq \frac{3\sqrt{3}}{8}$
   (c) $a^2 + b^2 + c^2 \geq 4K\sqrt{3}$ (Weitzenböck's inequality)

9. Let $p, q > 1$ be real numbers satisfying $1/p + 1/q = 1$.
   (a) Let $a, b > 0$. Prove Young's inequality: $ab \leq a^p/p + b^q/q$.
   (b) Let $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ be arbitrary reals. Prove Hölder's inequality:
$$\sum_{k=1}^{n} |x_k y_k| \leq \left(\sum_{k=1}^{n} |x_k|^p\right)^{1/p} \left(\sum_{k=1}^{n} |y_k|^q\right)^{1/q}$$

10. Let $p_1, p_2, \ldots, p_n$ and $q_1, q_2, \ldots, q_n$ be positive real numbers with
$$p_1 + p_2 + \cdots + p_n = q_1 + q_2 + \cdots + q_n = 1.$$
Prove Gibbs' inequality from information theory:
$$-\sum_{k=1}^{n} p_k \ln p_k \leq -\sum_{k=1}^{n} p_k \ln q_k.$$